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ABSTRACT

This paper studies the difficulties children have solving word problems that correspond to one of the situations modeled by multiplication: that of a ratio (e.g. "candies per box"), multiplied by an extensive quantity (e.g. "boxes") (I x E). Sixty children were individually observed solving and repeating I x E problems. Various tasks designed for this study were used to probe children's representation of the problem: drawings, model construction, and model recognition. Systematic errors in children's performance on these tasks were analyzed. The results of children's performance show that the major source of difficulty in solving I x E problems lies in a component of this process not common to all multiplication problems, understanding the concept of a ratio. This conclusion underscores the importance of conceptual understanding in solving word problems. The relevance of the study's results for teaching and research of word problems is discussed. (Author)

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CONCEPTUAL UNDERSTANDING IN SOLVING MULTIPLICATION WORD PROBLEMS

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Conceptual Understanding in Solving Multiplication Word Problems

It is widely acknowledged that often it is difficult for children to develop mathematical skills, particularly those related to word problem solving. Previous efforts to remedy this situation have not been marked by great success (see, for example, the 1979 National Assessment of Educational Progress report). Increasingly, it is argued that in order to improve educational practice we need a better understanding of the conceptual structure of the domains that we are asking children to master, and of the difficulties children have in developing this mastery. This work was undertaken with this perspective in mind and is intended to further the understanding of the source of children's difficulty in solving multiplication word problems.

Numerous studies have tried to determine the reasons for these difficulties. In the majority of these studies, however, the categories used for characterizing the word problems are very broad and frequently not informative (e.g. "Standardized word problems test", "Algebra word problems"). For understanding children's behavior in solving word problems there is a need to have a clear description of the types of problems that children are asked to solve.

This need has been recognized by a group of researchers (e.g. Suppes, P., Loftus, E.F. and Jerman, M.(1969); Loftus, E.F. (1970); Rosenthal, D. and Resnick (1971); Loftus, E.F. and Suppes, P. (1972); Jerman, M. and Dees, R.(1972); Jerman, M. (1972); Jerman, M. and Mirman, S. (1973); Segalla, A. (1973)). They attempt to determine what structural variable causes some problems to be more difficult than others. Some of the variables which are taken into account by this group are the

minimum number of operations required to reach the solution, the minimum number of steps required to reach the solution, the number of words in a problem, sequence of the information, verbal cues and unit conversion. Such variables are mainly syntactical.

To understand how children solve mathematics word problems, it is important to consider the semantic aspects of the problem; that is, the meaning of the concepts and relationship involved in the problem. The work of Carpenter, Hiebert and Moser (1979), Carpenter and Moser (1979), Greeno (1978), Moser (1979) and Nesher (1979) have shown this for one - step addition and subtraction word problems. They classified word problems into different semantic classes according to the situation that the operation models. They found that the problem's semantic class was a strong determiner of problem difficulty for the children. Now, multiplication and division word problems present new kinds of difficulties to the children. There is lack of research that looks at the role of semantic understanding in children's difficulty when solving multiplication word problems. This study works toward that end.

A typology for multiplication

This paper argued that understanding the concepts and relationships involved in a problem plays an important role in solving the problem. In order to make manifest the concepts and relationships present in different multiplication word problems it is important to distinguish among the situations that multiplication models. A typology of multiplication word problems developed by Schwartz (1976) is used as framework for this study.

Some definitions will be made to help describe the situations modeled by multiplication. In a mathematical context, numbers that are used in a sentence are used either as nouns or as adjectives. For example, in the sentence: "four

and three are seven", the numbers are used as nouns. This kind of number will be called nominal numbers. In the majority of cases in which people are asked to model situations which involve quantitative elements the number are used as adjectives. For example, "4 candies and 3 candies are seven candies". Numbers with referents will be called adjectival numbers.

In dealing with the multiplication of adjectival numbers; for example; "5 apples per dollar" times "3 dollars of apples", the nature of the quantities represented by "5" and the "3" are different. One of the quatities involve the notion of a ratio: "apples per dollar". Quantities of this type will be called intensive quantity. Intensive quantities are a generalization of the notion of density. On the other hand, there are quantities like "apples" "dollars" whose measure depends not on the "density" of the referent but in the total "amount". This type of quatities will be called extensive quantity.[1]

Having made these definitions, it can be said that there are five situations involving adjectival numbers which multiplication can model:

(i) (a) Extensive x quantifier (E x q)

like

12 candies x 3 times as many (candies)

(i) (b) Intensive x quantifier (E x q)

like

12 candies/bag, x 3 times as many (candies/bag)

(ii) Intensive x extensive (I x E)

like

12 candies/bag x 3 bags

(iii) Intensive x intensive (I x I)

like

10 candies/bag x 2 bags/child

(iv) Extensive x extensive (E x E)

like

3 kinds of blouses x 2 kinds of skirts

Main question

The main goal of this paper is to study the role of understanding the concepts and relationships involved in a problem in the process of solving it. For example, three of the situations modeled by multiplication involve intensive quantities. The works of Piaget and Inhelder (1941), Piaget, Grize, Szeminska and Vinh Bang (1968) and Strauss and Stavy (1979) show that the concept of intensity is difficult for children to understand. One would then expect this concept to be a source of difficulty in solving I x q, I x E and I x I problems. This work explores if this is the case for one of these structures.

Among the three structures involving intensive quantities, I x E was chosen for this study. This decision was based on the following analysis. The phrasing of the I x q problems involves the word "times" (e.g. 12 candies in each bag, 3 times as many). The use of this word might be a cue to children as how to solve this type of problem, even without understanding the situation. The I x q problem is not, then, appropriate for this study. Intuitively, I think that I x I problems are harder than I x E problems. For one thing, they involve the understanding of two intensive quantities. Also, the earlier appearance of I x E problems in the school curriculum implies that they are considered easier than I x I problems. Examining the students' textbooks (for grade 2, 3, 4, 5 and 6) of the series Mathematics for the Elementary School of the School Mathematics Study Group (1962), I found that more than 70 percent of the one-step multiplication problems were of the type I x E. Therefore, this work focuses on I x E problems as a first step in understanding children's

difficulty in solving and representing one-step multiplication problems.

Solving I x E problems of the following type: "There are 12 candies in each bag. María bought 4 bags of candies. How many candies will she have?" involve various components: Understanding the relationship between the two elements that enter to form an intensive quantity; that is, the relationship between the candies and the bags in the example above (this relationship will be denoted as R_1); understanding the relationship between the intensive and the extensive quantity (R_2); and mapping of the latter realtionship to a mathematical operation with the same structure (F).

In order to explore which of the components is the major source of difficulty, this study observed children solving and repeating I x E problems. Children also drew and constructed representations of the quantities and realtionships. They were asked to pick a correct representation among a series of drawings. An attempt was made to affect the difficulties of the problem by varying their content.

When content is changed in the I x E problems there are three factors that change: the extensive quantities (e.g. candies, box, tin cans, stomach); the intensive quantity (e.g. candies/box, tin cans/stomach); and the situation described in the problem (e.g. buying candies, eating tin cans). Among these factors the representation of the intensive quantity was expected to be the cause of major difficulty for the children. The intensive quantity was then varied along two dimensions: embeddedness[2] and visualization. The level of embeddedness depends on how familiar are different types of intensive quantities to the children. In terms of visualization the relationships expressed in intensive quantities whose elements are discrete quantities can be characterized as spatial, where the relationship can be pictured as it is given in the problem (e.g. candies/box); abstract (e.g. apples/dollar); and temporal (e.g. runs/game).

The results dealing with repetitions, representations and solution of I x E problems are expected to shed light on which of the components of the I x E problems can account for the difficulties that children experience with them. The fact that the difficulty of the problems will or will not vary with the level of embeddedness and/or visualization of the intensive quantity will bear on how important is the representation of the intensive quantity for the understanding of the problem.

The presence of errors in the representations of the intensive quantity (R_1) or/and of the relationship between the intensive and extensive quantities (R_2) in the drawings, model construction, model recognition and repetitions indicates that the representation of these relationships is a possible source of difficulty for the I x E problems. On the other hand if the relationships are represented correctly, but still children solve the problem incorrectly, then the difficulty must lie in mapping the structure of the problem to a mathematical operation with the same structure (F).

Experiment 1

Method

Subjects

The experiment was carried out with 36 children between the ages of 8 and 12 years. The children were students of the Longefellow (N=24) and Fletcher (N=12) Schools in Cambridge, Massachusetts. There were 20 boys and 16 girls. All the children had an Hispanic background. They all could speak Spanish but some of them preferred to read English and so the problems were presented in their language of preference.

Procedure and materials

Each child had two sessions with the researcher. In the first session, the children played a game for approximately 20 minutes in order to establish rapport. At the end of the game the researcher tried to extract contents that were familiar to the children through an informal conversation so as to design the problems for the next session.

The second time, around a week later, the experimenter worked two tasks with one child at a time. The session was conducted in Spanish and was audio-taped.

Task I : Problems

In this task the children solved a total of ten problems; six involving intensive quantities and four fillers (simple addition problems). The task was presented as a game in which the child accumulated points according to the numbers in the answers of the problems.

The problems involving intensive quantities were divided in two groups. Group I consisted of three similarly embedded problems problems which had different levels of visualization. These problems varied from child to child and were constructed in accordance with the content of familiarity of each child.

Group II consisted of three problems with similar levels of visualization (all were spatial) but with different levels of embeddedness. These problems were the same (except for the numbers) for all children (see Appendix A)

Half of the children solved first the problems of group I, consisting of similarly embedded problems, and half of the children solved first the problems of group II, consisting of problems with the same visualization. The order of the individual problems was permuted so that all possible combinations of order and number complexity were given. After each problem was solved, the experimenter

read the problem again and asked the child to repeat it.

Task II : Drawings

Once the children had worked all the problems they were asked to draw a picture or a sketch which corresponded to each of the I x E problems.

The first 18 children were asked to draw a picture or a picture or a sketch of the problem in such a way that if another student came to the room he could work the problem from the picture.

The last 18 children were asked specifically to represent the quantities in the problem. For example, if the problem was:

"In a store they sell 22 candies in each bag.
María bought 4 bags of candies.
How many candies will she have?"

the researcher asked the child: "draw 22 candies in each bag and 4 bags of candies".

Results

Fillers

The fillers were worked correctly[3] almost all of the times (142/152 (93.4%)). The fillers were used to identify children who had adopted some fixed solution strategy regardless of the actual problem. Two children were identified doing so. Both worked a filler problem according to the way they had previously worked the problem before it. For example, one (12 years, M) after working correctly three multiplication problems solved the following simple problem by multiplying (7 x 10):

"In a game between the Yankees and the Red Sox, the Yankees scored 7 runs and the Red Sox scored 10 runs. How many runs were scored in the game?"

These children were clearly not constructing independent solutions for every problem and so their answers were dropped from the analysis of the results. The results then show that 138 of the 144 fillers (95.8%) were correctly solved.

Performance levels

Difference in embeddedness

The performance of the children in the problems with the same type of visualization but different embeddedness correspond to our expectations. The problem involving tin cans per stomach, which is less embedded than candies per box, and children per room caused most difficulties ($p \leq .005$).

Difference in visualization

In case of similarly embedded problems with different visualization the results did not correspond to our expectations. The difference between problems is statistically non-significant. Surprinsingly enough there was a trend for the abstract problem to be the easiest to solve.

Drawings

Relation Between Drawings and Performance Level

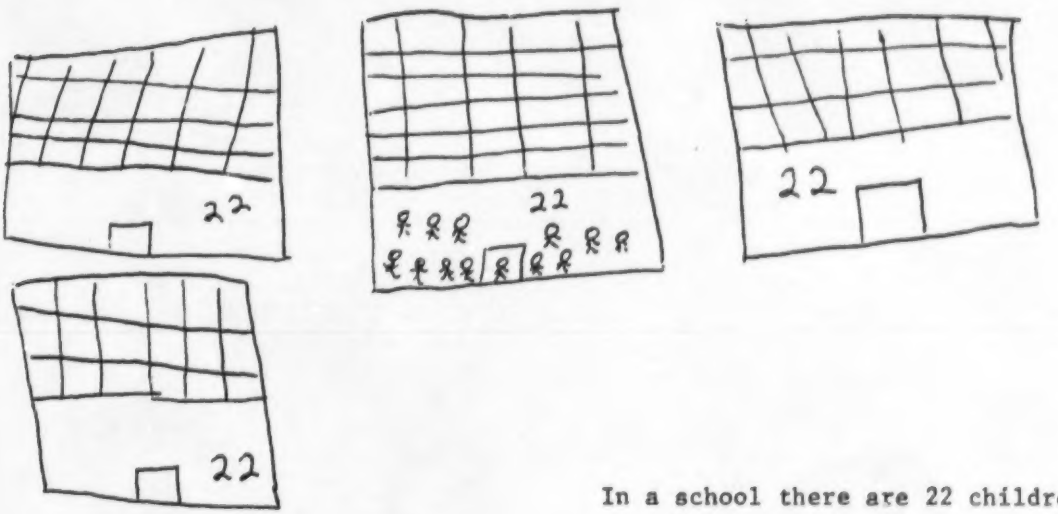
Children found the drawing task more difficult than the actual solution of the problems.

In fact, some (n=2) children completely refused to draw and others (n=3) refused to make drawings of some of the problems. Sixty-one problems correctly solved were incorrectly drawn while only 15 problems correctly drawn were incorrectly solved. This is quite surprising, given that the solving of the problems was done before the drawings and the problems incorrectly solved (n=85) were all concretely modeled in a way that could have been drawn. Yet only 15 of these models became the basis of correct drawings.

Types of drawings

There were two instructions for the drawings. The first 18 children (Group 1) were asked simply to make a picture of the situation, while the last 18 children (Group 2) were asked more directly to draw the quantities and relationships involved. In both groups we found the following types of drawings:

(1) Drawings which show the mathematical structure correctly:



In a school there are 22 children[4]
in each room.
The school has 4 rooms.
How many children are there in
the school?

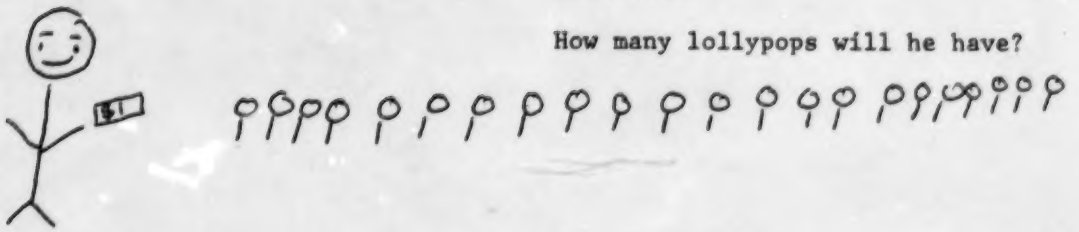
(10, F, correctly)[5]

(2) Drawings which an unprecise idea of both the intensive quantity and the mathematical structure:

There were 22 candies in
each box.
María bought 4 boxes of
candies.
How many candies will
she have?

(3) Representation of the intensive quantity only:

In a store they sell 22 lollypops for a dollar.
For a party John bought 4 dollars of lollypops.
How many lollypops will he have?



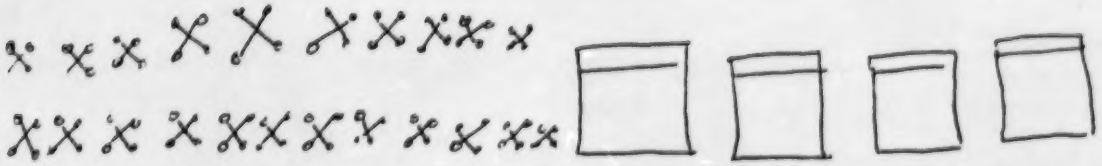
(12,M, correctly)

(4) Drawings which show either one or two of the quantities mentioned in the problems but not the relationship:

There were 22 candies
in each box.
María bought 4 boxes of
candies.
How many candies will she have?

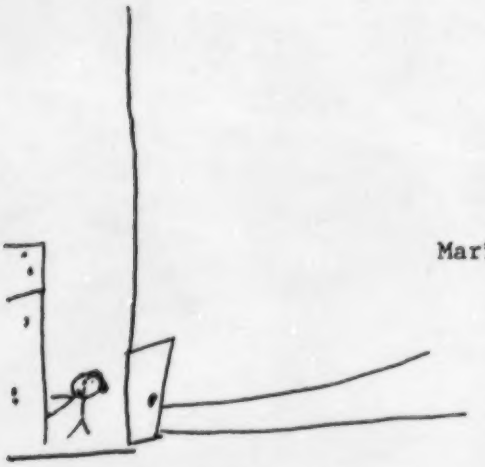
(10,F, incorrectly)

or



(11,F, Incorrectly)

(5) And children who only drew the situation in which the problem was given, but didn't draw any of the relevant quantities:



There were 22 candies in each box.
María bought 4 boxes of candies.
How many candies will she have.

María in the store

(11,M, incorrectly)

As can be seen from Table 1, both groups made drawings of the different types of problems discussed above. The explicit instruction given to Group II increased the proportion of correct drawings and almost eliminated purely situational drawings.

Insert Table 1 About Here

More important, Table 1 shows that when errors are made in the drawings of the quantities, almost all of the errors involved failure to represent the relationship between the elements that enter to from an intensive quantity; that is, the relation between candies and boxes in our example.

Repetitions

Relation between repetitions and performance

A close relationship between doing the problem correctly and repeating the problem correctly is found. Only 6 times out of 124 (4.8%) in which a child worked a problem incorrectly could he repeat it correctly.

Type of errors in repetitions

The errors in repetitions for both groups I and II were analyzed; children in group I were only asked to repeat the problems they had worked incorrectly. Fourteen responses in which either only the situation is described or it is not clear if the repetition correct or incorrect, are not included.

The errors can be divided into two broad categories: omission errors, in which the child leaves out one or both premises of the problem; and distortion errors, where some part of the problem is changed. Distortion was more common tham omission errors (see table 2). More important, out of the 51 distortion errors, 43(84%) consisted in a changed from an intensive to an extensive quantity. In the ommision errors, where one of the premises is not repeated, 5 times out of 6 the premise that is missing is the one involving the intensive quantity.

Insert Table 2 about here

Discussion

The first conclusion to be drawn fron this study is that the representation of the relationship between the elements that enter to form an intensive quantity (R_1) is the major stumbling block in being able to solve these problems. In the drawings, 103 f the 109 errors (93.6%) involved failure to represent the intensive quantity.

Repetitions , if not done by rote, are a way to restate the problem as the child has understood it. In this study two observations support this assertion: there is a high correlation between problems worked and repeated correctly, and children rarely repeated the problem in the same words as it was read to them, even when they repeated it correctly. So children's repetitions should reflect children's representations of the problems.

In the repetitions, in both the distortion (84%) and omission errors (83.4%), the major source of difficulty is also repeating the intensive quantity.

One experimental manipulation made was to vary the familiarity of the intensive quantity (called here degree of embeddedness). I found that the more embedded the intensive quantity was in the child's interest and experience, the better the performance.

This result supports the hypothesis that the representation of the intensive quantity is particularly important to solve these problems, for it was only the embeddedness of the intensive quantity that was varied. I made an effort to see that the quantities entering into the intensive relation were familiar (e.g. tin cans, stomach); even though the children would not have been expected to have manipulated ratios of tin cans per stomach before.

The degree of visualization of the intensive quantities was another variable in the study. The intensive quantities used in the study can be characterized as spatial, abstract, and temporal.

The spatial problems should have been easier to visualize since both the quantities and the relationships can be pictured as they were given in the problems; for example, candies per bag. It was expected then that they should have also been the easiest to solve. The results, however, show no significant difference among the three types of problems. Moreover, contrary to what was expected there was a trend for the abstract problems to be the easiest.

All the abstract problems in the study involved intensive quantities of the form, "x per dollar". The activity of finding how many things can be bought when the price is x per dollar, is more common than, for example, finding how many lolly pops in bags when there are x lolly pops per bag. In fact, it was easier for the researcher to find familiar situations involving an intensive quantity of the form

N-per dorfal "than finding one involving a spatial intensive quantity. It is possible then that the similarly embedded problems were not really so, being the abstract porblems the most embedded ones. This supports again the importance of embeddness.

The result of this experiment point to the representation of R_1 as the basic obstacle in being able to solve these problems. To arrive at this conclusion much weght was given to the representations of R_1 made by the children through drawings. Now, drawings was a harder task for the children than solving the problem. It could be then that the difficulty in drawing R_1 mainly lies in having an idea of how to represent it in a drawing and not in understanding the relationship. Therefore, it is necessary to probe by other means the representation that children make of the relationships presented in the problem.

Experiment II

Method

Subjects

The subjects were 24 children (13 girls, 11 boys) between 7 and 13 years of age. All the children have an Hispanic background. They all could speak Spanish but some of them preferred to read in English. The problems were presented in their language of preference.

Procedure and Materials

The tasks were undertaken with one child at a time. The whole group of tasks lasted one half hour. The experiment was divided into four parts. In the first part, the children worked the pre-test. In the second part, they worked two tasks which required them to represent the quantities and relationships involved in the I x E problems. In the third part, problems which were presented in verbal form in the pre-test were presented using concrete materials. Giving a concrete representation of the problem minimized the difficulties of representing the problem. It could then be tested if the children could map this representation of the problem's

situation to a mathematical operation that had the same structure. At the end of the experiment , they worked a post-test which was equivalent to the pre-test. Each problem was changed slightly in numbers and content, trying to maintain the same level of difficulty.

Pre-test

In the pre-test, each child solved a total of six problems: four I x E problems, and two fillers (an addition and a subtraction problem).

The I x E problems consisted of two problems involving spatial intensive quantities and two with abstract intensive quantities. The two spatial problems used the phrase "in each" to express the relationship R_1, while the abstract problem used the word "per".

Both the spatial problems and the abstract problems differed in degree of embeddedness, as follows:

Insert Table 3 about here.

The order of presentation of the problems was randomized for each subject. After each problem was solved, the experimenter read the problem again and asked teh child to repeat it.

Task 1: Model Construction

The children were asked to represent with concrete materials the situation described in each of the problems they had worked. For example, in the candies/box problem they were asked: "If these are boxes and these are candies, what would it be like if there are 4 boxes of candies and five candies in each box?" In the case of a filler I asked" "Suppose that I have five marbles and you gave me three additional marbles, what would that be like?"

Task 2: Model Recognition

In this task, the children were asked to choose among a set of drawings the one that represented the situation described to them by the experimenter. There was a question corresponding to each of the problems in the pre-test. For

each I x E problems there were seven drawings. Four of them had a representation of a relationship between the intensive and extensive quantities. One of them was the correct representation, while the rest incorrectly represented, for example, either the number of tin cans in each stomach or the number of stomachs; or both. The other two represented either one or both of the elements that enter to form the intensive relationship but not the relationship (see Figure 1). The experimenter displayed the drawings in front of the child and then asked, fro example: "If these are stomachs and theee are tin cans, which one among the drawings shows, 3 stomachs and 9 tin cans in each stomach". In both tasks (model construction and model recognition) the question was posed using the same locution ("in each" or "per") as in the problem.

Insert Figure 1 about here.

In the case of the fillers, there were five drawings. One represented the situation correctly; for example, 10 marbles plus 4 marbles, while the rest corresponded to the four drawings representing a relationship between an intensive and an extensive quantity used in the I x E porblems (see Figure 2).

Insert Figure 2 about here.

Task 3: Map from Concrete Model to Operation

The children were presented with concrete models of the situations corresponding to the different problems. For example, in the case of candies/box they were prsented with four containers with five candies each. They were then asked: "how many candies are there?"

Results

Pre-test

Fillers

The fillers were worked correctly almost all of the times (98.1%). Some children (N = 7) who added in all the problems were dropped from the study.

Performance level

Embeddness made a difference in the ability of the children to work the problems correctly. Considering the group of children who worked some problems correctly and others incorrectly, level of embeddness significantly affected performance (Chi Square Test, Spatial Problems, $p \leq .001$, Abstract $p \leq .01$).

Differences in Visualization-locution

The intensive relationship R_1 was expressed differently in the spatial and in the abstract problems. "In each" was used in the spatial problems, and 'per' in the abstract ones. The level of visualization is confounded with the use of the locutions 'in each' and 'per'.

The combination of these two variables, which will be denoted as visualization-locution, did affect the ability of the children to work the problems correctly. Considering the children who worked some but not all of the problems correctly, level of visualization, together with the use of a different locution to express R_1, significantly affected performance. (Chi Square Test, Highly Embedded problems $p \leq .01$, Low embedded $p \leq .001$)

Repetition

A close relationship between being able to do the problem correctly and being able to repeat the problem correctly is found. Only 42 out of 302 (14%) in which a child worked a problem incorrectly could he repeat it correctly.

Type of errors in repetition

The type of errors and the pattern of distribution of errors found in this experiment are similar to the ones found in Experiment I. The distortion errors are most common (80.3%), and the most frequent distortion is the change of intensive to an extensive quantity (74.5%).

Model Construction

Relation between model construction and performance level

Children found the model construction task easier than solving the problems (Wilcoxon Test $p \leq .01$). Seventy-three percent of the times solving the problem and the construction of the model were both correct or both incorrect. The fillers were modeled correctly almost all of the times (47 times out of 48).

Types of models

There were four types of models constructed. For example, in the problem: "there are 3 boxes of candies and 5 candies in each box", there were the following types of models:

(1) Models which show the mathematical structure correctly (62.5%)

(2) Representation of the intensive quantity only (2%)

(3) Models which show one or two of the quantities mentioned in the problem but not the relationship (27%)

(4) Models which interpret the relationship as "5 candies in 3 boxes" (8.3%)

The most common error was failure to represent correctly the relation between the elements that enter to from an intensive quantity (R_1).

Model recognition

The model recognition task was easier than the problems (Wilcoxon Test, $p \leq .01$). Seventy-two percent of the times, solving the problem and model recognition were both correct or both incorrect. The models representing the fillers were picked correctly almost all of the times. (47 times out of 48).

Types of representation picked by the children

Of the seven types of representation (see figure 2) only four were ever chosen: the correct one; the one representing correctly only the intensive quantity; the one representing only one element; and the one representing two of the elements but not the relationships.

The representations picked out by the children were very similar, both in type and in number, to the ones constructed. Again, the most common error was the failure to represent the relatioship between the elements that enter to form the intensive quantity.

Mapping from concrete model to operation

The mapping from the concrete model to the operation was not a source of difficulty. No matter how they performed on the pre-test, 24 of the 25 children, when presented with a model such as 4 containers with 5 candies each, could answer the question: "how many candies are there?

Post-test

The level of performance of some (N = 9) children in the post-test was different from that of the pre-test. Some children (N = 8) improved their performance, while one did worse. Considering the magnitude of change for each child, children significantly improved their performance (Wicoxon Test $p \leq .05$).

Discussion

The results of this experiment lend additional support to the major conclusion drawn from a previous experiment; that the represenattion of the intensive quantity is the major source of difficulty for solving I x E problems.

Repeating the problem incorrectly was closely related to working the problem incorrectly. Furthermore, the most common error ocurred when repeating the intensive quantity.

The degree of difficulty in solving the problems varied in accordance with the difficulty of the intensive quantity. Level of embeddness of the intensive quantity played a role in children's performance solving the problems. The combination of a different level of visualization with a different locution to express the intensive relatioship also affected the level of performance. The role that each of the variables; visualization and locution is playing can't be determined from the results. Another study should keep apart these two variables to determine the specific role that each one is playing. Two new tasks were introduced to probe children's representation of the problem: model construction and model recognition. Both tasks were easier for the children than solving the problem. The type of representation children made in the model construction task, and picked in the model recognition task were similar to one another. Moreover, the proportion of children

making, or picking, each type of represen tation were similar in both tasks. In the model construction and model recognit ion tasks, children only had to concentrate on representing the relationships involve d in the I x E problems, since the elements were either concretely present or illustr ated. In the model recognition task, the relationships were also represented so ch ildren only had to pick the correct one. Children's inability to construct or pick a correct representation of these relationships constitutes very strong evidence fo r the assertion that children are having difficulty in representing the relationsh ips. The similarity between the results from model construction and model recogni tion, in which they only had to pick a correct representation, attests to the deep nature of this difficulty.

In order to have a perspective o f the role of R_1 in solving I x E problems, the difficulties that children have with the other components of the process of solving I x E problems had to be examined .

Two relationships are involved i n representing I x E problems: one among the elements that enter to form the inten sive quantity (R_1), and another among the intensive and extensive quantity (R_2). I f children understood R_1 but not R_2, they should have been able to construct or pic t representations with a correct intensive quantity but which didn't reflect the str ucture of the problem. This rarely happened. It can be concluded, then, that in repres enting I x E problems, once children understand R_1, R_2 is not an additional s ource of difficulty.

Experiment II also studied child ren's performance in mapping a correct represntation of the problem's situation t o a mathematical operation with the same structure (this operation is designated F) . Only one child made an error in F, and it was just in one problem. This componen t is clearly not causing difficulty to the children in this age group.

The data of the experiment points to the component involving the understanding of the relationship between the elements t hat enter to form an intensive quantity (R_1) as the major source of difficulty in repre senting I x E problems. It could be argued that understanding R_1 is not needed to sol ve I x E problems, but that the under-

standing of R_1 and the ability to solve correctly I x E problems develop at the same time although independently of each other. Evidence against this asertion lies in the improvement of children's performance in this experiment. In the tasks of this experiment, children dealt with R_1, R_2, and F. Their improvement is an indication of their having learned doing these taks. Data has been presented to show that F is not a source of difficulty. Some evidence also shows that neither is R_2. So children's learning must consist basically in understanding R_1. This understanding then is the basic agent in improving children's performance.

The results of a control study rule out the possibility that their improvement was due to other factors like being more relaxed or gaining experience with the problems.

Conclusions

Children's major difficluty in representing I x E problems is representing the relationship between the elements that enter to form the intensive quantity (R_1). The results of a series of different tasks given to children: repetition, drawings, model construction, and model recognition all point to that conclusion. The improvement of children's performance after working with tasks which involved R_1 shows that being able to understand this relationship is a step in solving correctly the I x E problems.

Two other components of the process of solving I x E problems: representing the relationship between the intensive and extensive quantity, and mapping this represenattion to a mathematical operation with the same structure did not constitute sources of major difficulty for children between seven and thirteen years of age. So the key component in solving the I x E problems in this age group is understanding R_1.

This conclusion estabishes the importance of conceptual understanding in the process of solving multiplication word problems. The major difficulty that children had with I x E problems was not with the problem solving strategies, not even with a general component of the process of solving multiplication word problems. Rather the difficulty lies in being able to understand the relationship between the elements that enter to form an intensive quantity.

The concept of intensive quantity is not common to all types of multiplication word problems. So in order to understand children's difficulty in solving multiplication word problems, it is important to have precise typology to characterize the word problems that children are asked to solve.

This typology should be precise. If the categories used to characterize the problems are too broad, they will probably include problems with various structures. Therefore, when looking for a general cause of difficulty one may overlook the concepts needed to solve a problem, and so leave out an important source of difficulty. For example, if this study had looked at one-step multiplication problems without looking at the different situations which multiplication models, it may have overlooked the main type of difficulty with the I x E problems; that is, the lack of understanding of the relationship between the elements that enter to form the intensive quantity. Clearly, the difficulty children will have with other types of problems such as E x E problems is of another nature.

This study can be expanded by studying how children work one-step multiplication problems of different structures. If the difficulties that children encouter working these problems differ in systematic ways, relating these differences to the structure of the problem, will lend additional weight to the importance of a semantic typology of word problems as a framework for undersatnding how children work this type of problems.

This typology can also serve as a framawork fro school instruction. If children's difficulty in solving problems of different structures arise from representing concepts and relationships specific to the problem being solved, then helping children understand the situations modeled by multiplication should be an objective in teaching multiplication. The improvement of children's performance solving I x E problems after working with concrete materials and drawings suggests that the use of drawings and concrete materials to represent a word problems can be effective instructional aid.

Notes

(1) For a more detailed analysis of intensive and extensive quantity see Schwartz (1976).

(2) Following M. Donaldson (1978), we define embedded thinking as one which operates in the supportive content of meaningful events. Embedded thinking is given when a person is reasoning about familiar material, with which he has experienced a variety of relationships and, hence, finds it easier to generate and assess the connection between the facts.

(3) In this and in all of our analysis, "correct" means that the problem is posed correctly, there may be a computational error.

(4) She started drawing twenty two children in each rrom. I then suggested to her to write "22". Few children wrote the number on their own initiative.

(5) (10, F, correctly) means: the drawing was done by a 10 years old girl who worked the problem correctly.

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Appendix A Example of Problems in Experiment I

Group I Similarly embedded problems

Spanish Version	English Version
Spatial Problems	
Marisol guarda 12 trajes de muñeca en cada caja.	Mary keeps 12 dolls' dresses in each box.
Ella tiene 4 cajas de trajes de muñeca.	She has 4 boxes of dolls' dresses.
¿Cuántos trajes de muñeca ella tiene?	How many dolls' dresses does she have?
En la tiendita venden 22 chicles por bolsa.	In a store they sell 22 bubble gums per bag.
Pedro compró 6 bolsas de chicles.	Peter bought 6 bags of bubble gum.
¿Cuántos chicles él compró?	How many bubble gums did he buy?
Abstract Problems	
En la tiendita venden los dulces a 22 dulces por peseta.	In a store they sell 22 candies per quarter.
Minerva compró 4 pesetas de dulces.	For a party Joan bought 4 quarters of candies.
¿Cuántos dulces compró?	How many candies will she have?
En la tiendita venden 22 paletas por un peso.	In a store they sell 22 lollypops for a dollar.
Para una fiesta Juan compró 6 pesos de paleta.	For a party John bought 4 dollars of lollypops.
¿Cuántas paletas él compró?	How many lollypops will he have?

Temporal Problems

En una serie de pelota el equipo de Juan bateó 12 hits por juego.	In a baseball series John's team batted 22 hits per game.
El equipo jugó 4 juegos en la serie.	In the series they played in 6 games.
¿Cuántos hits batearon en toda la serie?	How many hits did they bat in the series?

Group II Similar Level of Visualization

Spanish Version	English Version
Cada cajita de dulces trae 22 dulces.	Each box of candies brings 22 candies.
María compró 4 cajitas.	María bought 4 boxes of candies.
¿Cuántos dulces ella tendrá?	How many candies will she have?
En una escuela hay 22 niños por salón.	In a school there are 22 children per room.
La escuela tiene 6 salones.	The school has 6 rooms.
¿Cuántos niños hay en la escuela?	How many children are there in the school?
Quizás tu sabes que la comida favorita de los fantasmas son las latas.	As you may know, ghosts' favorite food are tin cans.
Ellos pueden acomodar 12 latas en cada estómago.	They can fit 12 tin cans in each stomach.
Héctor es un fantasma que tiene 4 estómagos.	Héctor is a ghost who has 4 stomachs.
¿Cuántas latas puede comer Héctor?	How many tin cans can Héctor eat?

Table 1

Types of drawings

	correct	Unprecise: intensive quantity and mathematical structure	Intensive quantity only	One or two of the quantities not the relationship	Situacion only
Group I	26.8%	6.4%	5.3%	41.9%	19.3%
Group II	45.3%	6.1%	2%	45.3%	1%

Table 2

Type of error when repeating problems

		Number of times
Omission Errors	Forgets whole problem	6
	Missing one premiss: A. Premiss involving intensive quantity	5
	B. Premiss involving extensive quantity	1
Distortion Errors	Changes intensive to extensive Example: From 22 candies/box, 4 boxes to 22 boxes, 4 boxes	43
	Changes relationship from one of multiplication to one of division Example: From 22 candies/box, 4 boxes to 22 candies in 4 boxes	5
	Changes extensive quantity Example: From 22 candies/box, 4 boxes to 22 candies/box, 4 candies	3

Table 3

Type of I x E problems in Experiment 2

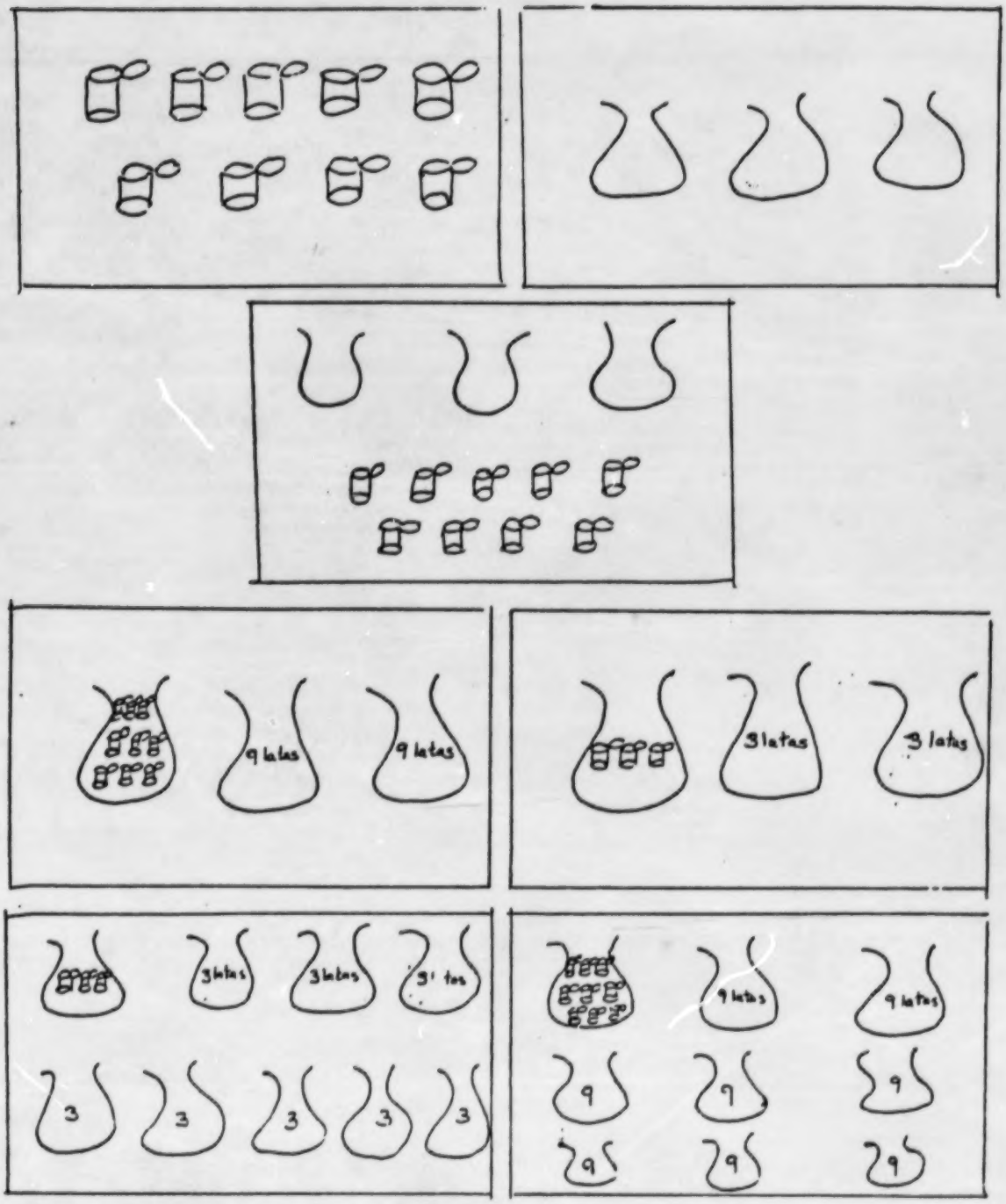
	Spatial	Abstract
Highly embedded	S_1: Candies/box	a_1 Apples/dollars
Low embeddedness	S_2: Tin cans/stomach	a_2 Patients/doctors

Type of error when repeating problems

(Experiments I, II and III)

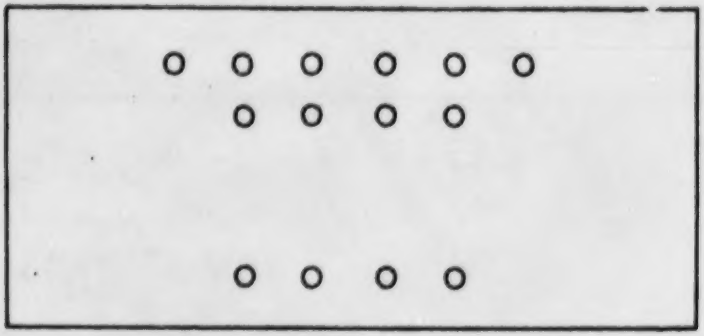
	Types of error	Number of times
Omission Errors	Forgets whole problem	4
	Missing one premise: A. Premise involving intensive quantity	16
	B. Premise involving extensive quantity	7
	Total	27
Distortion Errors	Changes intensive to extensive Example: From 22 candies/box 4 boxes; to 22 boxes, 4 boxes	82
	Changes relationship from one of multiplication to one of division Example: From 22 candies/box, 4 boxes; to 22 candies in 4 boxes	4
	Changes extensive quantity Example: From 22 candies/box 4 boxes; to 22 candies/box, 4 candies	24
	Total	110

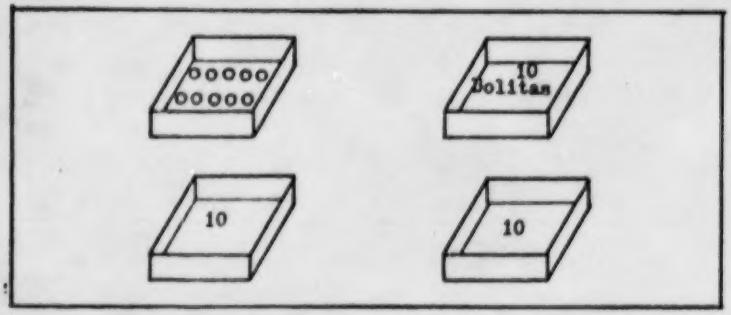
Figure 1: Drawings for the Model Recognition Task, I x E Problems

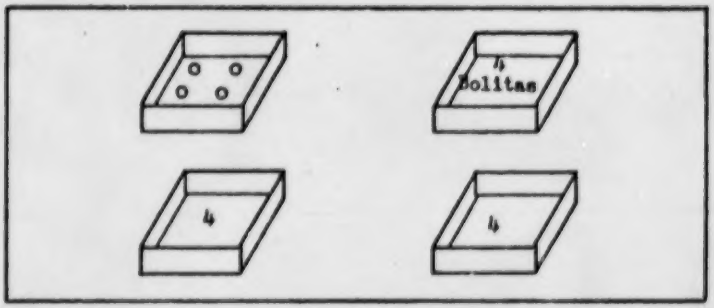


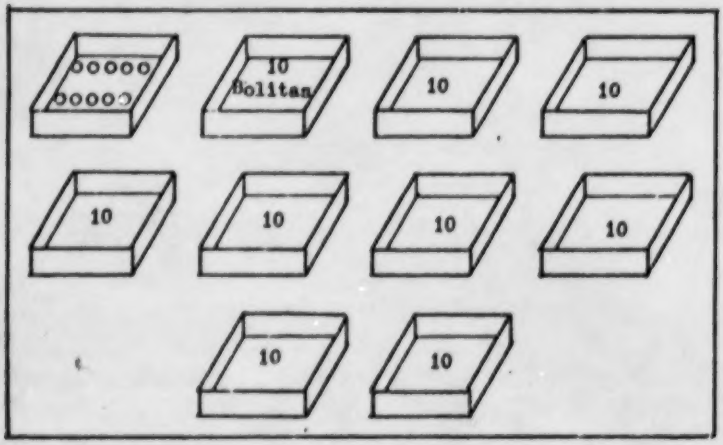
Problem: If these are stomachs and these are tin cans, which one among the drawings shows 3 stomachs and 9 tin cans in each stomach?

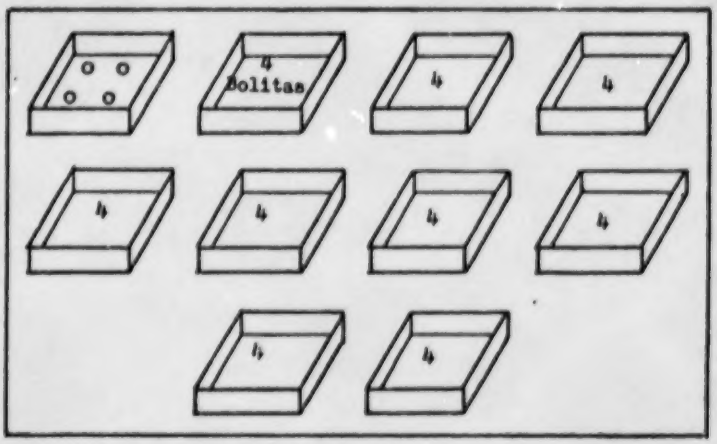
Figure 5: Drawings for the Model Recognition Task, Fillers











Problem: If these are marbles and these are boxes, which one among the drawings shows 10 marbles and 4 additional marbles?